Breather solutions to the focusing nonlinear Schrödinger equation

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The N-breather solution to the focusing nonlinear Schrödinger equation is presented. It is shown that the breather is linearly unstable, but the unstable modes are overstabilized and do not destroy the structure of the breather. It is also demonstrated that the breather solution can be constructed as an imbricate series of rational growing-and-decaying modes. [S1063-651X(98)01903-5]

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I. INTRODUCTION

The self-modulation of one-dimensional waves in a nonlinear dispersive medium can be described by the nonlinear Schrödinger (NLS) equation

$$iu_t+u_{xx}+q|u|^2u=0, \tag{1}$$

which has been derived in various branches of physics [1–5]. It is well known that if $q<0$, a plane wave is stable for the modulation and if $q>0$, the plane wave is not stable but subject to the modulational instability. Equation (1) with $q>0$ is called the focusing NLS (FNLS) equation and has an N-envelope-soliton solution which satisfies the boundary condition $u\to 0$ as $|x|\to\infty$ [6,7]. On the other hand, Eq. (1) with $q<0$ is called the defocusing NLS (DNLS) equation and has the N-dark-hole soliton solution which satisfies the boundary condition $|u|^2\to$const as $|x|\to\infty$ and which was given by Hirota [8] as

$$u=\rho_0 \exp(i\theta)\,\frac{g}{f}, \tag{2}$$

where

$$f=\sum_{\mu=0,1}\exp\left[\sum_{i>j}^{(N)}A_{ij}\mu_i\mu_j+\sum_{j=1}^{N}\mu_j\eta_j\right],$$

$$g=\sum_{\mu=0,1}\exp\left[\sum_{i>j}^{(N)}A_{ij}\mu_i\mu_j+\sum_{j=1}^{N}\mu_j(\eta_j+2i\phi_j)\right], \tag{3}$$

and

$$\exp(A_{ij})=\left[\frac{\sin\frac{1}{2}(\phi_i-\phi_j)}{\sin\frac{1}{2}(\phi_i+\phi_j)}\right]^2,$$

$$\eta_j=p_jx-\Omega_jt+\eta_j^0,$$

$$p_j^2=-2q\rho_0^2\sin^2\phi_j,$$

$$\Omega_j=2kp_j-p_j^2\cot\phi_j,$$

$$\theta=kx-\omega t,$$

$$\omega=k^2-q\rho_0^2,$$

where ϕ_j are distinct real constants, $\Sigma_{\mu=0,1}$ is the summation over all possible combinations of $\mu_1=0,1$, $\mu_2=0,1,\ldots$, $\mu_N=0,1$, and $\Sigma_{i>j}^{(N)}$ indicates the summation over all possible pairs chosen from N elements, and η_j^0 are arbitrary phases.

Substituting the expression (2) into the NLS equation (1), we have the coupled equations for f and g,

$$(iD_t+2ikD_x+D_x^2)g\cdot f=0,$$

$$(D_x^2+q\rho_0^2)f\cdot f-q\rho_0^2|g|^2=0, \tag{4}$$

where we have used the boundary condition $|u|^2\to\rho_0^2$ as $|x|\to\infty$ and $\theta=kx-\omega t$, $\omega=k^2-q\rho_0^2$, the operators D_t, D_x, and various products of them are defined by

$$D_t^n a\cdot b\equiv\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n a(t)b(t')|_{t'=t},$$

$$D_x^n a\cdot b\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n a(x)b(x')|_{x'=x}.$$

We see that the bilinear form (4) of the DNLS (FNLS) equation is converted into that of the FNLS (DNLS) equation under the transformation

$$x\to ix,\quad t\to -t,\quad k\to -ik,\quad \omega\to -\omega,\quad q\to -q. \tag{5}$$

This fact shows the possibility that the solution of the DNLS equation with expression (2) can be transformed into the solution of the FNLS equation by using the transformation (5). Ablowitz and Herbst [9] have already shown that the $2N$-dark-hole soliton solution [8] of the DNLS equation becomes the solution of the FNLS equation under the transformation (5) with $k=0$ provided the evenness condition, $u(x,t)=u(-x,t)$, is satisfied.

It is well known that an exact periodic solution to the soliton equations can often be expressed by the sum of constituents which have a localized structure individually, such as solitons [10–12]. Such a superposition was found by Toda [13] for the case of the cnoidal wave of the Toda lattice and the Korteweg–de Vries (KdV) equations. Zaitsev [14] and Tajiri and Murakami [15] have succeeded in obtaining the periodic soliton solution and the lattice soliton solution by the nonlinear superposition of the rational soliton solutions for the Kadmtsev-Petviashvili (KP) equation with positive dispersion, respectively. Recently, it was shown that the nonlinear periodic wave solutions to the Boussinesq equation can be constructed as the imbricate series of rational growing-and-decaying modes which are localized in space and time [16,17]. The uniform state with negative background described by the Boussinesq equation is linearly unstable for all waves. The rational growing-and-decaying mode solution can be in existence on such a uniform state. Taking into account the fact that the uniform plane wave described by the FNLS equation is not stable but subject to the modulational instability [18–20], we can expect that the FNLS equation also has the rational growing-and-decaying mode solution, which solution has already been found by Akhmediev, Eleonskii, and Kulagin [21]. And we can also expect that overstabilized wave solutions to the FNLS equation are constructed as the imbricate series of the rational growing-and-decaying modes.

In this paper we discuss the solutions of the FNLS equation. The purposes of this study are to show that (i) the FNLS equation has the N-breather solution, (ii) the recurrent wave solutions can be constructed as the imbricate series of rational growing-and-decaying modes, and (iii) the breather solution is linearly unstable but the linearly unstable modes do not destroy the structure of the breather.

II. BREATHER SOLUTIONS

We consider periodic envelope-wave solutions of the FNLS equation

$$iu_t+u_{xx}+q|u|^2u=0 \quad (q>0), \tag{6}$$

with the boundary condition

$$|u|^2\to\rho_0^2 \quad \text{as} \quad |x|\to\infty. \tag{7}$$

Under the transformation (5), the $2N$-dark-hole soliton solution of the DNLS equation $q<0$ transforms the solution of the FNLS equation $q>0$ with

$$p_l=i\sqrt{2q}\rho_0 \sin \phi_l,$$

$$p_{l+M}=-i\sqrt{2q}\rho_0 \sin \phi_l \quad (l=1,2,\dots,M),$$

$$\omega=k^2-q\rho_0^2 \tag{8}$$

for $N=2M$ (M is an integer), which has been pointed out by Ablowitz and Herbst [9]. It should be noted that p_j ($j=1,2,\dots,N$) are distinct pure imaginary and ϕ_j and Ω_j ($j=1,2,\dots,N$) are real constants. In this paper we call it the N-growing-and-decaying mode solution.

We now consider the extension of the solution to the case that p_j, Ω_j, and ϕ_j are complex numbers. First of all, we confirm that the two-soliton solution with complex wave numbers and frequencies satisfies Eq. (6). The two-soliton solution may be written as

$$u=\rho_0 \exp(i\theta)\,\frac{g}{f}, \tag{9}$$

where

$$f=1+e^{\eta_1}+e^{\eta_2}+ae^{\eta_1+\eta_2},$$

$$g=1+e^{\eta_1+2i\phi_1}+e^{\eta_2+2i\phi_2}+ae^{\eta_1+\eta_2+2i(\phi_1+\phi_2)}, \tag{10}$$

and

$$\eta_j=P_jx-\Omega_jt+\eta_j^0 \quad (j=1,2),$$

$$\theta=kx-\omega t,$$

where P_j, Ω_j, η_j^0, and ϕ_j are complex. Substituting Eq. (10) into Eq. (4) with $q>0$, we find that if the following relations are satisfied, Eq. (9) is the solution of Eq. (6),

$$\omega=k^2-q\rho_0^2,$$

$$\phi_2=\phi_1^*\pm\pi,$$

$$P_1=i\sqrt{2q}\rho_0 \sin \phi_1,$$

$$P_2=i\sqrt{2q}\rho_0 \sin \phi_2=-i\sqrt{2q}\rho_0 \sin \phi_1^*,$$

$$\Omega_j=2kP_j-P_j^2 \cot \phi_j \quad (j=1,2),$$

$$a=\left[\frac{\sin \frac{1}{2}(\phi_1-\phi_2)}{\sin \frac{1}{2}(\phi_1+\phi_2)}\right]^2=\left[\frac{\cos \frac{1}{2}(\phi_1-\phi_1^*)}{\cos \frac{1}{2}(\phi_1+\phi_1^*)}\right]^2. \tag{11}$$

Then, we have the breather solution

$$u=\rho_0 \cos 2\phi_R e^{i(\theta+2\phi_R)}\Bigg[1+\frac{1}{\sqrt{a}\cosh(\eta_R+\sigma)+\cos \eta_I} \times\left\{\left(\frac{\cosh 2\phi_I}{\cos 2\phi_R}-1\right)\cos \eta_I+i\left(\tan 2\phi_R \sinh(\eta_R+\sigma) -\frac{\sinh 2\phi_I}{\cos 2\phi_R}\sin \eta_I\right)\right\}\Bigg], \tag{12}$$

where $\eta_R=P_Rx-\Omega_Rt+\eta_R^0$, $\eta_I=P_Ix-\Omega_It+\eta_I^0$, and σ is a constant, and $P_1=P_R+iP_I$, $\Omega_1=\Omega_R+i\Omega_I$, and a are determined for given $\phi_1=\phi_R+i\phi_I$ by Eq. (11) as follows:

$$(P_R^2-P_I^2)+q\rho_0^2(1-\cos 2\phi_R \cosh 2\phi_I)=0,$$

$$2P_RP_I+q\rho_0^2 \sin 2\phi_R \sinh 2\phi_I=0,$$

$$\Omega_R-2kP_R+\frac{(P_R^2-P_I^2)\sin 2\phi_R+2P_RP_I \sinh 2\phi_I}{\cosh 2\phi_I-\cos 2\phi_R}=0,$$

$$\Omega_I-2kP_I-\frac{(P_R^2-P_I^2)\sinh 2\phi_I-2P_RP_I \sin 2\phi_R}{\cosh 2\phi_I-\cos 2\phi_R}=0,$$

$$a=\frac{\cosh^2 \phi_I}{\cos^2 \phi_R}. \tag{13}$$

The condition of the nonsingular solution, $a>1$, is always satisfied. Figure 1 shows the typical time development of the breather solution.

As special cases of this solution, (i) the solution with $\phi_R\neq0$ and $\phi_I=0$ corresponds to the first homoclinic orbit

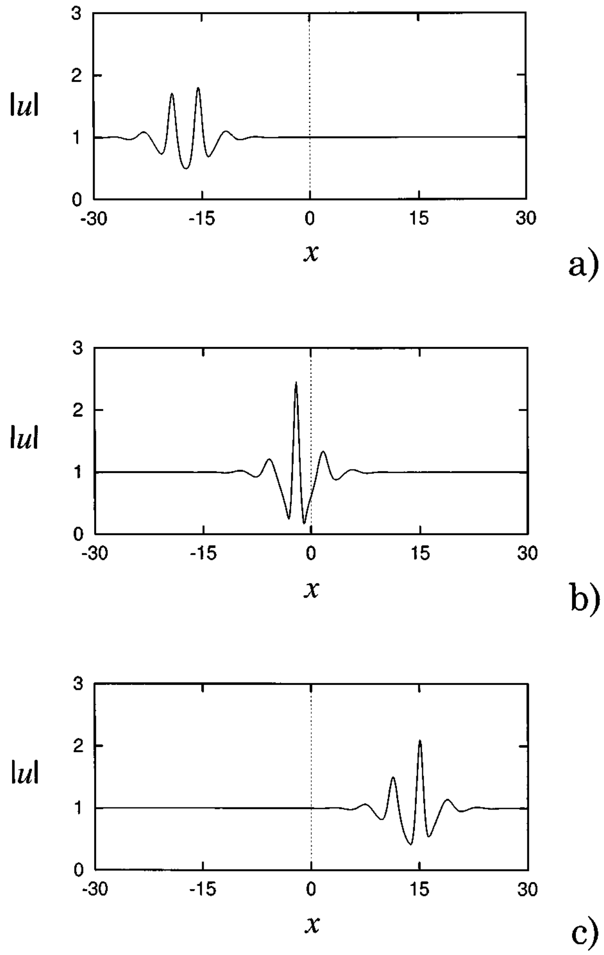

FIG. 1. Typical time development of a breather solution for $q=2$, where x and u are dimensionless.

obtained by Ablowitz and Herbst [9], which is periodic in the x direction and localized in time. We call this solution the growing-and-decaying mode solution hereafter in this paper. (ii) The solution with $k=0$, $\phi_R=0$, and $\phi_I\neq 0$ corresponds to the stationary breather solution, which is periodic in time and localized in the x space. (iii) Taking $\phi_R=\epsilon\gamma$ and $\phi_I=\epsilon\delta$, we have

$$P_R=-\sqrt{2q}\rho_0\delta\epsilon+O(\epsilon^3),$$

$$P_I=\sqrt{2q}\rho_0\gamma\epsilon+O(\epsilon^3),$$

$$\Omega_R=(2q\rho_0^2\gamma-2k\rho_0\sqrt{2q}\delta)\epsilon+O(\epsilon^3),$$

$$\Omega_I=(2q\rho_0^2\delta+2k\rho_0\sqrt{2q}\gamma)\epsilon+O(\epsilon^3),$$

$$\sqrt{a}=1+\tfrac{1}{2}(\gamma^2+\delta^2)\epsilon^2, \tag{14}$$

and

$$f=[(\tilde{\eta}_R^2+\tilde{\eta}_I^2)+(\gamma^2+\delta^2)]\epsilon^2+O(\epsilon^3),$$

$$g=[(\tilde{\eta}_R^2+\tilde{\eta}_I^2)-3(\gamma^2+\delta^2)+4i(\gamma\tilde{\eta}_R+\delta\tilde{\eta}_I)]\epsilon^2+O(\epsilon^3), \tag{15}$$

as $\epsilon\to 0$, where $\eta_R-\eta_R^0=\epsilon\tilde{\eta}_R+O(\epsilon^2)$ and $\eta_I-\eta_I^0=\epsilon\tilde{\eta}_I+O(\epsilon^2)$. Substituting Eq. (15) into Eq. (9) and taking the limit $\epsilon\to 0$, we have

$$u=\rho_0\,\exp(i\theta)\left(1-\frac{4+8iq\rho_0^2t}{1+2q\rho_0^2(x-2kt)^2+4q^2\rho_0^4t^2}\right). \tag{16}$$

This is an exact solution which is localized in space and time as shown in Fig. 2 and we call this solution the rational growing-and-decaying mode.

It is interesting to note that even if we take the wave number P_j and frequency Ω_j complex ($j=1,2$), the dispersion relation (11) is the same form as the case where P_j are pure imaginary and Ω_j are real. The same statements are true for four-soliton solutions. This suggests that the N-breather solution to the FNLS equation (6) can be expressed as

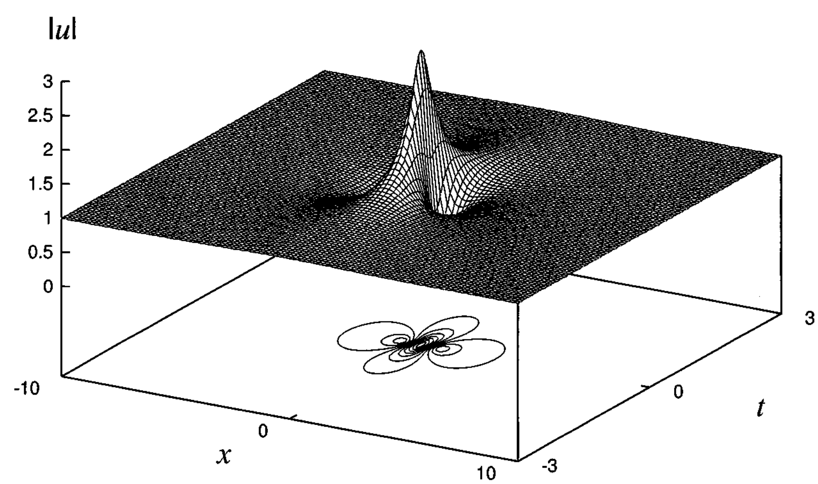

FIG. 2. Rational growing-and-decaying mode solution of the FNLS equation with $q=2$ for $\rho_0=1$ and $k=0$. Curved lines drawn at the bottom of this figure are contour lines. In this figure, x, t, and u are all dimensionless.

$$u=\rho_0 \exp(i\theta) \frac{g}{f}, \tag{17}$$

$$f=\sum_{\mu=0,1} \exp\left[\sum_{i>j}^{(2N)} A_{ij}\mu_i\mu_j+\sum_{j=1}^{2N} \mu_j\eta_j\right],$$

$$g=\sum_{\mu=0,1} \exp\left[\sum_{i>j}^{(2N)} A_{ij}\mu_i\mu_j+\sum_{j=1}^{2N} \mu_j(\eta_j+2i\phi_j)\right], \tag{18}$$

where

$$\eta_j=P_jx-\Omega_jt+\eta_j^0,$$

$$\eta_{n+N}=\eta_n^*,$$

$$P_{n+N}=P_n^*,$$

$$\Omega_{n+N}=\Omega_n^*,$$

$$\phi_{n+N}=\phi_n^*+\pi$$

for $j=1,\ldots,2N$, $n=1,2,\ldots,N$;

$$\omega=k^2-q\rho_0^2,$$

$$P_j=i\sqrt{2q}\rho_0 \sin \phi_j,$$

$$\Omega_j=2kP_j-P_j^2 \cot \phi_j \tag{19}$$

for $j=1,2,\ldots,2N$; and

$$\exp(A_{mn})=\left[\frac{\sin \frac{1}{2}(\phi_m-\phi_n)}{\sin\frac{1}{2}(\phi_m+\phi_n)}\right]^2,$$

$$\exp(A_{m,n+N})=\left[\frac{\cos \frac{1}{2}(\phi_m-\phi_n^*)}{\cos \frac{1}{2}(\phi_m+\phi_n^*)}\right]^2,$$

$$\exp(A_{m+N,n+N})=\left[\frac{\sin \frac{1}{2}(\phi_m^*-\phi_n^*)}{\sin \frac{1}{2}(\phi_m^*+\phi_n^*)}\right]^2 \tag{20}$$

for $m=1,2,\ldots,N$, $n=1,2,\ldots,N$, where ϕ_n are distinct complex constants. Here, it should be noted that in the DNLS equation we cannot take the wave numbers and frequencies, P_j and Ω_j, complex since the solution becomes singular.

III. BREATHER SOLUTION AS IMBRICATE SERIES OF RATIONAL GROWING-AND-DECAYING MODES

The purpose of this section is to show that the growing-and-decaying mode, stationary breather, and breather solutions can be constructed by the imbricate series of the rational growing-and-decaying modes.

A. Growing-and-decaying mode

First of all, we try to construct the growing-and-decaying mode solution. It is interesting to note that the rational growing-and-decaying mode solution (16) is rewritten as the following form:

$$u=\rho_0 \exp\{i[kx-(k^2-q\rho_0^2)t]\} \times\left(1+\frac{1}{iq\rho_0^2t+\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}}\right) \times\left(1+\frac{1}{iq\rho_0^2t-\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}}\right). \tag{21}$$

Especially in the case where the rational growing-and-decaying mode does not propagate, i.e., $k=0$, it takes the following form:

$$u=\rho_0 \exp(iq\rho_0^2t)\left(1+\frac{1}{iq\rho_0^2t+\frac{1}{2}\sqrt{1+2q\rho_0^2x^2}}\right) \times\left(1+\frac{1}{iq\rho_0^2t-\frac{1}{2}\sqrt{1+2q\rho_0^2x^2}}\right). \tag{22}$$

On the basis of Eq. (22), we assume the form of a growing-and-decaying mode solution with $k=0$ as follows:

$$u=\rho_0 \exp[i(\sigma t+\phi)]\left(1+b\sum_n \frac{1}{i\alpha t+\nu(x)+n}\right) \times\left(1+b\sum_{n'} \frac{1}{i\alpha t-\nu(x)+n'}\right), \tag{23}$$

where the summation Σ_n means $\lim_{N\to\infty} \Sigma_{n=-N}^{N}$, $\nu(x)$ is a function of x to be determined, and α and σ are real constants. This expression shows the superposition of the rational growing-and-decaying mode about the x direction because the parts connected with x are treated as real functions and the parts connected with t are treated as pure imaginary functions. We note that Eq. (23) is rewritten as

$$u=\rho_0 \exp[i(\sigma t+\phi)](1+b\pi \cot\{\pi[\nu(x)+i\alpha t]\}) \times(1-b\pi \cot\{\pi[\nu(x)-i\alpha t]\}). \tag{24}$$

After substituting Eq. (24) into Eq. (6), we have

$$\sigma=q\rho_0^2(1+\pi^2b^2)^2, \tag{25}$$

$$\left(\frac{d\nu(x)}{dx}\right)^2=\frac{q\rho_0^2b^2}{2}[1-\pi^2b^2 \cot^2 2\pi\nu(x)], \tag{26}$$

$$\frac{d^2\nu(x)}{dx^2}=q\rho_0^2\pi^3b^4 \cot 2\pi\nu(x)\left[\left(\frac{1+2\pi^2b^2}{\pi^2b^2}-\frac{\alpha}{q\rho_0^2\pi^2b^3}\right) +\cot^2 2\pi\nu(x)\right]. \tag{27}$$

Comparing Eq. (27) with the derivation of Eq. (26) by x and using the relation $1+\cot^2 A=\operatorname{cosec}^2 A$ we find

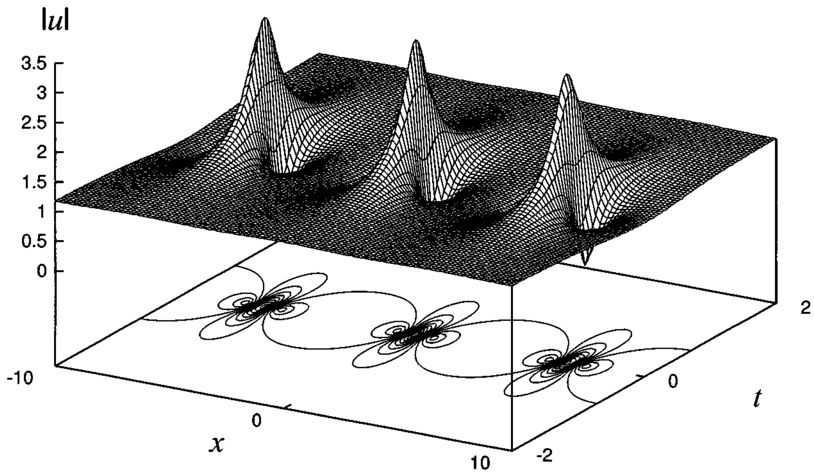

FIG. 3. Growing-and-decaying mode solution of the FNLS equation with $q=2$ for $\rho_0=1$, $b=0.137$, $C=0$, and $k=0$. As we can see, it is constructed as the imbricate series of rational growing-and-decaying modes in the x direction.

$$\alpha=q\rho_0^2(1+\pi^2b^2)b. \tag{28}$$

Integrating Eq. (26), we obtain the form of $\nu(x)$,

$$\nu(x)=\frac{1}{2\pi}\arccos\left[\frac{1}{\sqrt{1+\pi^2b^2}}\cos(\sqrt{2\pi^2\alpha b}x+C)\right]. \tag{29}$$

These show that Eq. (23) is the solution to Eq. (6) if two constants, σ and α, and the function $\nu(x)$ are given by Eqs. (25), (28), and (29), respectively. From Eqs. (23), (25), (28), and (29), we obtain an exact solution,

$$\begin{aligned}u=&\rho_0(1+\pi^2b^2)\exp\{i[q\rho_0^2(1+\pi^2b^2)^2t+\phi]\}\\&\times\left[1-\frac{2\pi b}{1+\pi^2b^2}\right.\\&\left.\times\frac{\pi b\cosh 2\pi\alpha t+i\sinh 2\pi\alpha t}{\cosh 2\pi\alpha t-(1/\sqrt{1+\pi^2b^2})\cos(\sqrt{2\pi^2\alpha b}x+C)}\right].\end{aligned} \tag{30}$$

This solution is periodic in the x direction and it grows exponentially at initial stage from the time-oscillate background, takes the maximum amplitude at a time, and finally decays exponentially to the time-oscillate background, which we call the growing-and-decaying mode solution. In the case $q=2$, Eq. (30) is in agreement with the solution shown by Ablowitz and Herbst. A typical growing-and-decaying mode solution is shown in Fig. 3. Comparing Figs. 2 and 3 is helpful for us to understand that the growing-and-decaying mode solution can be constructed as the imbricate series of rational growing-and-decaying modes.

Since Eq. (6) is invariant under the Galilei transformations,

$$\begin{aligned}x'&=x-2kt,\\t'&=t,\\u(x',t')&=\exp[-i(kx+k^2t)]u(x,t),\end{aligned} \tag{31}$$

we find that the growing-and-decaying mode solution with $k\neq 0$ is constructed by the following imbricate series:

$$\begin{aligned}u=&\rho_0\exp(i\{kx+[-k^2+q\rho_0^2(1+\pi^2b^2)^2]t\})\\&\times\left(1+b\sum_n\frac{1}{i\alpha t+\nu(x-2kt)+n}\right)\\&\times\left(1+b\sum_{n'}\frac{1}{i\alpha t-\nu(x-2kt)+n'}\right).\end{aligned} \tag{32}$$

The imbricate series of rational growing-and-decaying modes, Eq. (23) or Eq. (32), is modified from the usual way of applying imbricate series which is the superposition of the whole solitary wave. It is very interesting to note that the growing-and-decaying mode is constructed by the products of two imbricate series.

B. Stationary breather

Next, in the same way as in the preceding section, we construct another periodic solution, the stationary breather solution, from the superposition of the rational growing-and-decaying modes. On the basis of Eq. (22), we assume the form of a stationary breather solution as

$$\begin{aligned}u=&\rho_0\exp[i(\zeta t+\phi)]\left(1+ih\sum_n\frac{1}{\kappa t+i\mu(x)+n}\right)\\&\times\left(1+ih\sum_{n'}\frac{1}{\kappa t-i\mu(x)+n'}\right),\end{aligned} \tag{33}$$

where the summation means $\lim_{N\to\infty}\Sigma_{n=-N}^{N}$, $\mu(x)$ is a function of x to be determined, and, κ and ζ are real constants. This expression shows the superposition of the rational growing-and-decaying modes about the t direction because the parts connected with t are treated as real functions and the parts connected with x are treated as pure imaginary functions. Rewriting Eq. (33) as

$$u=\rho_0 \exp[i(\zeta t+\phi)](1+h\pi \coth\{\pi[\mu(x)-i\kappa t]\})\times(1-h\pi \coth\{\pi[\mu(x)+i\kappa t]\}) \tag{34}$$

and substituting Eq. (34) into Eq. (6), we have

$$\zeta=q\rho_0^2(1-\pi^2h^2)^2, \tag{35}$$

$$\left(\frac{d\mu(x)}{dx}\right)^2=\frac{q\rho_0^2h^2}{2}[1-\pi^2h^2 \coth^2 2\pi\mu(x)], \tag{36}$$

$$\frac{d^2\mu(x)}{dx^2}=q\rho_0^2\pi^3h^4 \coth 2\pi\mu(x)\left[\left(\frac{1-2\pi^2h^2}{\pi^2h^2}+\frac{\kappa}{q\rho_0^2\pi^2h^3}\right)+\coth^2 2\pi\mu(x)\right]. \tag{37}$$

Comparing Eq. (37) with the derivation of Eq. (36) by x and using the relation $\coth^2 A-1=\mathrm{cosech}^2 A$ gives

$$\kappa=-q\rho_0^2(1-\pi^2h^2)h. \tag{38}$$

Integrating Eq. (36), we obtain the form of $\mu(x)$,

$$\mu(x)=\frac{1}{2\pi}\operatorname{arccosh}\left[\frac{1}{\sqrt{1-\pi^2h^2}}\cosh(\sqrt{-2\pi^2\kappa h}x+C)\right]. \tag{39}$$

From Eqs. (33), (35), (38), and (39), we obtain the following exact solution:

$$u=\rho_0(1-\pi^2h^2)\exp[iq\rho_0^2(1-\pi^2h^2)^2t+i\phi]\left[1+\frac{2\pi h}{1-\pi^2h^2}\times\frac{\pi h \cos 2\pi\kappa t-i \sin 2\pi\kappa t}{\cos 2\pi\kappa t-(1/\sqrt{1-\pi^2h^2})\cosh(\sqrt{-2\pi^2\kappa h}x+C)}\right]. \tag{40}$$

This is the stationary breather solution. This solution is localized in space and grows and decays recurrently in time-oscillate background.

C. Breather solution

It is difficult to construct the breather solution in the same way as in the previous two sections. It is interesting to note that the absolute square of the breather solution is given by

$$|u|^2=\rho_0^2+\frac{2}{q}\frac{\partial^2}{\partial x^2}\ln f, \tag{41}$$

with

$$f=1+2e^{\eta_R}\cos \eta_I+ae^{2\eta_R}, \tag{42}$$

where

$$\eta_R=P_Rx-\Omega_Rt+\eta_R^0,$$

$$\eta_I=P_Ix-\Omega_It+\eta_I^0,$$

and a and relations among $P_R, P_I, \Omega_R, \Omega_I$ are given by Eq. (13). And furthermore, taking into account that u is unchanged even if f is multiplied by $\exp(ax+b)$ with a and b independent of x, we find that the breather solution can be expressed by

$$|u|^2=\rho_0^2+\frac{2}{q}\frac{\partial^2}{\partial x^2}\ln[\sqrt{a}\cosh(P_Rx-\Omega_Rt+\sigma)-\cos(P_Ix-\Omega_It+\theta)], \tag{43}$$

where $\sigma=\eta_R^0+\frac{1}{2}\ln a$, $\theta=\eta_I^0+\pi$.

In Fig. 4, this solution is drawn for a particular choice of the constants. The time development of the breather appears to be complicated at one view as shown in Fig. 1. However, when we depict the breather solution in the x-t plane, it seems to represent an inclined sequence of rational growing-and-decaying modes as shown in Fig. 4. This leads to the conjecture that the breather solution is also expressed by the imbricate series of rational growing- and-decaying modes. But, if we assume the same expression for the breather solution as the previous sections, the calculation becomes tedious. We note that the rational growing-and-decaying mode solution (21) can be rewritten as

$$|u|^2=\rho_0^2-\frac{1}{q}\frac{\partial^2}{\partial x^2}\ln\left[\frac{1}{[\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}+iq\rho_0^2t]^2}\times\frac{1}{[\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}-iq\rho_0^2t]^2}\right]. \tag{44}$$

Then, we assume the form of the imbricate series for the breather solution as follows:

$$|u|^2=\rho_0^2-\frac{1}{q}\frac{\partial^2}{\partial x^2}\ln\left\{\left[\sum_{n=-\infty}^{\infty}\frac{1}{[\varphi(x,t)-i\psi(x,t)-n]^2}\right]\times\left[\sum_{n=-\infty}^{\infty}\frac{1}{[\varphi(x,t)+i\psi(x,t)-n]^2}\right]\right\}, \tag{45}$$

where $\varphi(x,t)$ and $\psi(x,t)$ are functions of x and t to be determined. It is important to note that Eq. (45) is rewritten in the form

$$|u|^2=\rho_0^2-\frac{1}{q}\frac{\partial^2}{\partial x^2}\ln\{[\pi^2\mathrm{cosec}^2\pi(\varphi-i\psi)]\times[\pi^2\mathrm{cosec}^2\pi(\varphi+i\psi)]\}=\rho_0^2+\frac{2}{q}\frac{\partial^2}{\partial x^2}\ln[\cosh 2\pi\psi-\cos 2\pi\varphi]. \tag{46}$$

Comparing this equation with Eq. (43), we find

$$\cosh 2\pi\psi=\sqrt{a}\cosh(P_Rx-\Omega_Rt+\sigma),$$

$$\cos 2\pi\varphi=\cos(P_Ix-\Omega_It+\theta), \tag{47}$$

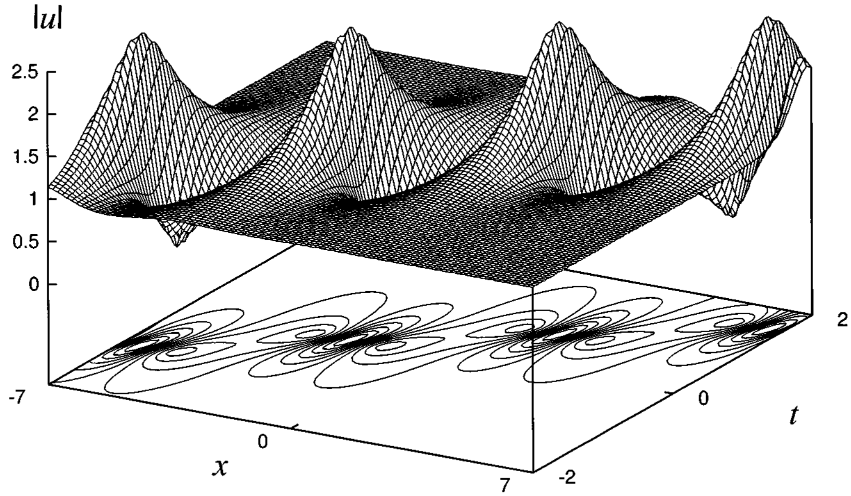

FIG. 4. Breather solution of the FNLS equation with $q=2$ for $\rho_0=1$, $P_R=0.5$, $P_I=1.5$, and $k=0.1$. We can see that the breather solution is constructed as an inclined sequence of rational growing-and-decaying modes to the x and t axes.

or

$$\cosh 2\pi\psi=\cosh(P_R x-\Omega_R t+\sigma),$$

$$\cos 2\pi\varphi=\frac{1}{\sqrt{a}}\cos(P_I x-\Omega_I t+\theta). \tag{48}$$

Equations (47) and (48) are readily solved to give

$$\psi=\frac{1}{2\pi}\ln[\sqrt{a}\cosh(P_R x-\Omega_R t+\sigma)+\sqrt{a\cosh^2(P_R x-\Omega_R t+\sigma)-1}],$$

$$\varphi=\frac{1}{2\pi}(P_I x-\Omega_I t+\theta), \tag{49}$$

and

$$\psi=\frac{1}{2\pi}(P_R x-\Omega_R t+\sigma),$$

$$\varphi=\frac{1}{2\pi}\arccos\left(\frac{1}{\sqrt{a}}\cos(P_I x-\Omega_I t+\theta)\right), \tag{50}$$

respectively. The substitution of Eq. (49) or Eq. (50) into Eq. (45) gives the breather solution as an imbricate series of rational growing-and-decaying modes.

Now, we consider the asymptotic formulas of these solutions. Taking the limit (a) $\phi_R\to 0$ and $\phi_I\to 0$ with $\phi_R/\phi_I\to 0 (P\to 0,\Omega\to 0)$, we have

$$\psi=\frac{1}{2\pi}\sqrt{1+2q\rho_0^2(x-2kt)^2}\,\phi_I,$$

$$\varphi=\frac{1}{2\pi}(2q\rho_0^2 t)\phi_I, \tag{51}$$

(b) $\phi_R\to 0$ and $\phi_I\to 0$ with $\phi_I/\phi_R\to 0$, we have

$$\psi=-\frac{1}{2\pi}(2q\rho_0^2 t)\phi_R,$$

$$\varphi=\frac{1}{2\pi}\sqrt{1+2q\rho_0^2(x-2kt)^2}\,\phi_R. \tag{52}$$

Thus for very small ϕ_R and ϕ_I (very small P and Ω), the solution (45) to Eq. (6) having ψ and φ defined by Eqs. (49) and (50) is approximated by

$$|u|^2=\rho_0^2-\frac{1}{q}\frac{\partial^2}{\partial x^2}\ln\left\{\left[\sum_{n=-\infty}^{\infty}\frac{1}{[\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}+iq\rho_0^2 t-\pi n/\phi_I]^2}\right]\times\left[\sum_{n=-\infty}^{\infty}\frac{1}{[\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}-iq\rho_0^2 t-\pi n/\phi_I]^2}\right]\right\} \tag{53}$$

and

$$|u|^2=\rho_0^2-\frac{1}{q}\frac{\partial^2}{\partial x^2}\ln\left\{\left[\sum_{n=-\infty}^{\infty}\frac{1}{[\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}+iq\rho_0^2 t+i\pi n/\phi_R]^2}\right]\times\left[\sum_{n=-\infty}^{\infty}\frac{1}{[\frac{1}{2}\sqrt{1+2q\rho_0^2(x-2kt)^2}-iq\rho_0^2 t+i\pi n/\phi_R]^2}\right]\right\}, \tag{54}$$

respectively; which are simple summations of rational growing-and-decaying mode solutions. The constituent hump of the breather with small P_R and P_I resembles rational growing-and-decaying mode sufficiently as shown in Fig. 4. In this sense, it is proper to regard the breather as the nonlinear superposition of rational growing-and-decaying modes.

IV. STABILITY OF BREATHER

In this section, the stability of the breather is studied by making use of the breather and the growing-and-decaying mode solution. The plane wave of Eq. (6),

$$u=\rho_0 \exp[i(kx-\omega t)], \tag{55}$$

with the dispersion relation

$$\omega=k^2-q\rho_0^2, \tag{56}$$

is linearly unstable to infinitesimal modulational perturbations of the form

$$u=\rho_0 \exp[i(kx-\omega t)]\left\{1+\hat{\epsilon}_+(t)\exp\left[ip\left(x-\frac{\partial\omega}{\partial k}t\right)\right]\right.$$
$$\left.+\hat{\epsilon}_-(t)\exp\left[-ip\left(x-\frac{\partial\omega}{\partial k}t\right)\right]\right\}, \tag{57}$$

with $\hat{\epsilon}_\pm(t)=\hat{\epsilon}_\pm(0)\exp(\sigma t)$, where the growth rate σ is given by

$$\sigma^2=p^2(2q\rho_0^2-p^2). \tag{58}$$

This is well known as the Benjamin-Feir instability [22]. The most peculiar feature of solutions to the NLS equation is the existence of the Fermi-Pasta-Ulam recurrence phenomenon in the long time evolution of the unstable solution. Lake *et al.* [18] have shown that the numerical solution of Eq. (6) with periodic boundary conditions and with a Benjamin-Feir unstable initial condition shows that a state of maximum modulation is reached by the unstable wave system and after reaching maximum modulation, the solution demodulates and eventually returns to an unmodulated state. It is interesting to note that the nonlinear evolution of an unstable mode is described by the growing-and-decaying mode solution which is given by Eq. (12) with $\phi_I=0$. The mode grows exponentially at initial stage ($t=-T'$, $T'\gg 1$) as follows:

$$u=\rho_0 \exp[i(kx-\omega t+4\phi_R)]\{1+\epsilon e^{\Omega_R t'}\cos(px-\Omega_I t+\eta_I^0)\}, \tag{59}$$

where $t'=t+T'$ and $\epsilon=(2/a)(e^{-2i\phi_R}-1)\exp(-\Omega_R T'-\eta_R^0)\ll 1$, takes the maximum modulation at a time, and finally ($t=T''\gg 1$) returns exponentially to the initial state as follows:

$$u=\rho_0 \exp[i(kx-\omega t)]\{1+\epsilon' e^{-\Omega_R t''}\cos(px-\Omega_I t+\eta_I^0)\}, \tag{60}$$

where $t''=t-T''$ and $\epsilon'=2(e^{2i\phi_R}-1)\exp(-\Omega_R T''+\eta_R^0)$ and the growth rate Ω_R and frequency Ω_I are given by

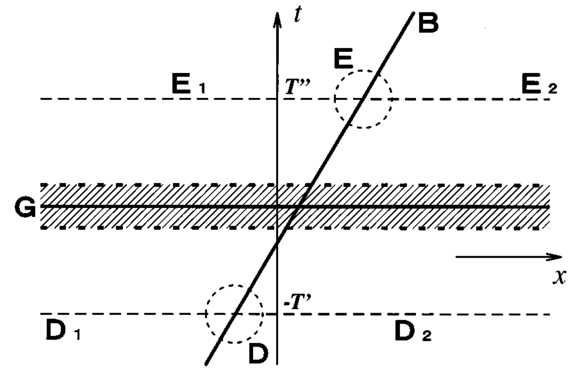

FIG. 5. The schematic diagram of the world lines of breather and growing-and-decaying mode. Breather takes the maximum amplitudes on line B. Growing-and-decaying mode takes the maximum amplitude on line G. In the neighborhood of line G, the growing-and-decaying mode is grown, but at regions far from line G, i.e., at $t=-T'$ and $t=T''$ ($T',T''\gg 1$), the amplitudes of the growing-and-decaying mode are too small, as at $t=-T'$ the growing-and-decaying mode does not grow yet, and at $t=T''$ it damps to die out already, respectively.

$$\Omega_R=p^2\cot\phi_R=p\sqrt{2q\rho_0^2-p^2},$$

$$\Omega_I=2kp=\frac{\partial\omega}{\partial k}p, \tag{61}$$

which are in agreement with the growth rate (58) and the frequency of modulational perturbation given by Eq. (57), respectively.

Now, we consider a solution consisting of a breather and growing-and-decaying mode. This is obtained from Eqs. (17)–(20), where $N=2$ is set and the breather wave number, frequency, and phase, $P_1=P_R+iP_I$, $\Omega_1=\Omega_R+i\Omega_I$, and $\phi_1=\phi_R+i\phi_I$, are taken as the same as Eq. (13) and the growing-and-decaying mode wave number and frequency, $P_2=ip=i\sqrt{2q}\rho_0\sin\phi_2$ and $\Omega_2=\Omega_{2R}+i\Omega_{2I}=p\sqrt{2q\rho_0^2-p^2}+i2kp$. The world lines of the breather and growing-and-decaying mode in the x-t plane are schematically drawn in Fig. 5. The line G shows the maximum amplitude of the growing-and-decaying mode. The growing-and-decaying mode has a finite value only in the shaded region near the line G. The breather and growing-and-decaying mode solution before growth of the growing-and-decaying mode is expressed approximately by the following equations: (i) in the region D in Fig. 5,

$$u=\rho_0 e^{i(kx-\omega t+4\phi_2)}\left[\frac{g_0}{f_0}+\hat{\epsilon}e^{\Omega_{2R}t'}\frac{g_1 f_0-f_1 g_0}{f_0^2}\right], \tag{62}$$

with

$$g_0=1+2e^{\eta_R+\sigma+2i\phi_R}[\cosh 2\phi_I\cos(\eta_I+\theta_1+\theta_2)$$
$$-i\sinh 2\phi_I\sin(\eta_I+\theta_1+\theta_2)]+M_1 e^{2\eta_R+2\sigma+4i\phi_R}, \tag{63}$$

$$g_1=e^{-2i\phi_2}[\cos p(x-2kt) + e^{\eta_R+2i\phi_R}(L_1\{\cosh 2\phi_I \cos[\eta_I+p(x-2kt)+\theta_1] - i \sinh 2\phi_I \sin[\eta_I+p(x-2kt)+\theta_1]\} + L_2\{\cosh 2\phi_I \cos[\eta_I-p(x-2kt)+\theta_2] - i \sinh 2\phi_I \sin[\eta_I-p(x-2kt)+\theta_2]\}) + M_1L_1L_2e^{2\eta_R+4i\phi_R} \cos[p(x-2kt)+\theta_1-\theta_2]], \quad (64)$$

$$f_0=1+2e^{\eta_R+\sigma}\cos(\eta_I+\theta_1+\theta_2)+M_2e^{2\eta_R+2\sigma}, \quad (65)$$

$$f_1=\cos p(x-2kt)+e^{\eta_R}\{L_1 \cos[\eta_I+p(x-2kt)+\theta_1] + L_2 \cos[\eta_I-p(x-2kt)+\theta_2]\} + M_1L_1L_2e^{2\eta_R}\cos[p(x-2kt)+\theta_1-\theta_2], \quad (66)$$

where

$$\sigma=\ln(L_1L_2),$$

$$\left[\frac{\cosh \phi_I}{\cos \phi_R}\right]^2=M_1, \quad \left[\frac{1}{\cos \phi_2}\right]^2=M_2,$$

$$\left[\frac{\sin\frac{1}{2}(\phi_1-\phi_2)}{\sin\frac{1}{2}(\phi_1+\phi_2)}\right]^2=L_1e^{i\theta_1}, \quad \left[\frac{\cos\frac{1}{2}(\phi_1-\phi_2)}{\cos\frac{1}{2}(\phi_1+\phi_2)}\right]^2=L_2e^{i\theta_2},$$

$$\hat{\epsilon}=\frac{2}{M_2}e^{-\Omega_{2R}T'-\eta_2^0},$$

$$\eta_R=P_Rx-\Omega_Rt+\eta_R^0, \quad \eta_I=P_Ix-\Omega_It+\eta_I^0,$$

and $t'=t+T'$; and (ii) in the regions D_1 and D_2, the solutions are given by

$$u=\rho_0e^{i(kx-\omega t+4\phi_2)}\{1+2L_1L_2e^{\eta_R} \times[(e^{2i\phi_R}\cosh 2\phi_I-1)\cos(\eta_I+\theta_1+\theta_2) - ie^{2i\phi_R}\sinh 2\phi_I \sin(\eta_I+\theta_1+\theta_2)] + \hat{\epsilon}(e^{-2i\phi_2}-1)e^{\Omega_{2R}t'}\cos[p(x-2kt)]\} \quad (67)$$

and

$$u=\rho_0e^{i(kx-\omega t)}\left\{1+\frac{2}{M_1L_1L_2}e^{-\eta_R} \times[(e^{-2i\phi_R}\cosh 2\phi_I-1)\cos(\eta_I+\theta_1+\theta_2) - ie^{-2i\phi_R}\sinh 2\phi_I\sin(\eta_I+\theta_1+\theta_2)] + \frac{\hat{\epsilon}}{L_1L_2}(e^{-2i\phi_2}-1)e^{\Omega_{2R}t'}\cos[p(x-2kt)+\theta_1-\theta_2]\right\}, \quad (68)$$

respectively. Here, we note that e^{η_R} is small and $e^{-\eta_R}$ is very small in Eqs. (67) and (68), respectively. Equations (62), (67), and (68) show that the perturbation of wave number p on the breather grows exponentially at initial stage with the same growth rate Ω_{2R} as Eq. (58) for the Benjamin-Feir instability. This means that the breather solution is linearly unstable to modulational perturbation. The perturbation reaches a state of maximum modulation on the line G in the x-t plane. After reaching maximum modulation, the perturbation begins to damp and then damps to die out at sufficiently large time as the following forms: (i) in the region E in Fig. 5,

$$u=ce^{i(kx-\omega t)}\left[\frac{\overline{g_0}}{\overline{f_0}}+\overline{\epsilon}e^{-\Omega_{2R}t''}\frac{\overline{g_1}\overline{f_0}-\overline{f_1}\overline{g_0}}{\overline{f_0}}\right], \quad (69)$$

with

$$\overline{g_0}=1+2e^{\eta_R+2i\phi_R}[\cosh 2\phi_I\cos\eta_I-i\sinh 2\phi_I\sin\eta_I] + M_1e^{2\eta_R+4i\phi_R}, \quad (70)$$

$$\overline{g_1}=e^{2i\phi_2}[\cos p(x-2kt) + e^{\eta_R+2i\phi_R}(L_1\{\cosh 2\phi_I \cos[\eta_I+p(x-2kt)+\theta_1] - i \sinh 2\phi_I \sin[\eta_I+p(x-2kt)+\theta_1]\} + L_2\{\cosh 2\phi_I \cos[\eta_I-p(x-2kt)+\theta_2] - i \sinh 2\phi_I \sin[\eta_I-p(x-2kt)+\theta_2]\}) + M_1L_1L_2e^{2\eta_R+4i\phi_R} \cos[p(x-2kt)+\theta_1-\theta_2]], \quad (71)$$

$$\overline{f_0}=1+2e^{\eta_R}\cos\eta_I+M_1e^{2\eta_R}, \quad (72)$$

$$\overline{f_1}=\cos p(x-2kt)+e^{\eta_R}\{L_1 \cos[\eta_I+p(x-2kt)+\theta_1] + L_2 \cos[\eta_I-p(x-2kt)+\theta_2]\} + M_1L_1L_2e^{2\eta_R}\cos[p(x-2kt)+\theta_1-\theta_2], \quad (73)$$

where $\overline{\epsilon}=2e^{-\Omega_{2R}T''+\eta_2^0}$, $t''=t-T''$; and (ii) in regions E_1 and E_2, the solutions are given by

$$u=\rho_0e^{i(kx-\omega t)}\{1+2e^{\eta_R}[(e^{2i\phi_R}\cosh 2\phi_I-1)\cos\eta_I - ie^{2i\phi_R}\sinh 2\phi_I\sin\eta_I] + \overline{\epsilon}(e^{2i\phi_2}-1)e^{-\Omega_{2R}t''}\cos[p(x-2kt)]\} \quad (74)$$

and

$$u=\rho_0e^{i(kx-\omega t)+4i\phi_R}\left\{1+\frac{2}{M_1}e^{-\eta_R}[(e^{-2i\phi_R}\cosh 2\phi_I-1) \times\cos\eta_I-ie^{-2i\phi_R}\sinh 2\phi_I\sin\eta_I] + \overline{\epsilon}L_1L_2(e^{2i\phi_2}-1)e^{-\Omega_{2R}t''}\cos[p(x-2kt)+\theta_1-\theta_2]\right\}. \quad (75)$$

We can obtain the one-breather and N-growing-and-decaying modes solution by using the bilinear form. The solution shows that the growing-and-decaying modes damp to die out at sufficiently large time and only the breather remains finally. The nonlinear development of these unstable modes on the breather are described by the solutions that superpose the one-breather and the N-growing-and-decaying modes. The linear unstable modes do not destroy the structure of the breather, in other words, the identity of the breather is not lost. The effect of the unstable modes on the breather is the phase shifts of the plane wave and the breather after enough time passed.

V. CONCLUSIONS

We have shown that the FNLS equation has the N-breather solutions with the boundary condition $|u|^2=\rho_0^2$ as $|x|\rightarrow\infty$. As special cases of the breather solution, we have the growing-and-decaying mode, stationary breather, and rational growing-and-decaying mode solutions. The growing-and-decaying mode solution is obtained by taking the limit $\phi\rightarrow 0$, which is localized in space and time. Here, we have to note that the breather solution to the DNLS equation becomes singular. It is also shown that the growing-and-decaying mode, stationary breather, and breather solutions can be constructed as the imbricate series of rational growing-and-decaying modes. In this sense, we can regard the rational growing-and-decaying mode as the constituent of recurrent wave solutions on the unstable wave field. Their periodic solutions are constructed by the products of two imbricate series, rather than a single imbricate series as in other applications. The stability character of the breather is investigated by using the exact solution. The breather is linearly unstable. The nonlinear development of unstable modes on the breather is described by the solution consisting of the one-breather and growing-and-decaying modes by using the bilinear form. It is shown that the linearly unstable modes are overstabilized and do not destroy the structure of the breather.

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